Breaking numbers

Lie Fu

University of Strasbourg

π-day/week, 2023

What is π ?

$\pi =$ the ratio of a circle's circumference to its diameter.

Approximately,

$$\pi = 3.14159265358979323846264338327950288419716939937510 \\ 58209749445923078164062862089986280348253421170679 82 \\ 148086513282306647093844 6\ldots$$

A great mathematician popularized the Greek letter π to denote this universal constant.

$\pi\epsilon\rho\acute{\iota}\mu\epsilon\tau\rho o\sigma$ ("perimetros")

Leonhard Euler (1707 - 1783)

In 1740, Philippe Naudé wrote to Euler asking :

Question : how many ways a number (=positive integer) can be written as a sum of distinct numbers ?

Euler's answer : Theory of partitions !

Partitions

Definition : A *partition* of a natural number is a way of writing it as sum of natural numbers.
Example : There are 7 different partitions of 5.

$$\begin{aligned} 5 &= 5 \\ 4+1 &= 5 \\ 3+2 &= 5 \\ 3+1+1 &= 5 \\ 2+2+1 &= 5 \\ 2+1+1+1 &= 5 \\ 1+1+1+1+1 &= 5 \end{aligned}$$

Odd and Distinct partitions

Definition : A partition is called *odd* if all parts are odd numbers. A partition is called *distinct* if all parts are distinct numbers.
Example : Among partitions of 5,

$$\begin{aligned} 5 &= 5 \\ 4+1 &= 5 \\ 3+2 &= 5 \\ 3+1+1 &= 5 \\ 2+2+1 &= 5 \\ 2+1+1+1 &= 5 \\ 1+1+1+1+1 &= 5 \end{aligned}$$

there are 3 distinct ones and 3 odd ones.

Denote by $D(n)$ the number of distinct partitions of n
Denote by $O(n)$ the number of odd partitions of n.

n	$D(n)$	$O(n)$
1	1	1
2	1	1
3	2	2
4	2	2
5	3	3
6	4	4
7	5	5
$\vdots$	$\vdots$	$\vdots$

$10 = 10$	$9 + 1 = 10$
$9 + 1 = 10$	$7 + 3 = 10$
$8 + 2 = 10$	$7 + 1 + 1 + 1 = 10$
$7 + 3 = 10$	$5 + 5 = 10$
$7 + 2 + 1 = 10$	$5 + 3 + 1 + 1 = 10$
$6 + 4 = 10$	$5 + 1 + 1 + 1 + 1 + 1 = 10$
$6 + 3 + 1 = 10$	$3 + 3 + 3 + 1 = 10$
$5 + 4 + 1 = 10$	$3 + 3 + 1 + 1 + 1 + 1 = 10$
$5 + 3 + 2 = 10$	$3 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 10$
$4 + 3 + 2 + 1 = 10$	$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 10$

$$D(10) = O(10) = 10$$

$11 = 11$	$11 = 11$
$10 + 1 = 11$	$9 + 1 + 1 = 11$
$9 + 2 = 11$	$7 + 3 + 1 = 11$
$8 + 3 = 11$	$7 + 1 + 1 + 1 + 1 = 11$
$8 + 2 + 1 = 11$	$5 + 5 + 1 = 11$
$7 + 4 = 11$	$5 + 3 + 3 = 11$
$7 + 3 + 1 = 11$	$5 + 3 + 1 + 1 + 1 = 11$
$6 + 5 = 11$	$5 + 1 + 1 + 1 + 1 + 1 + 1 = 11$
$6 + 4 + 1 = 11$	$3 + 3 + 3 + 1 + 1 = 11$
$6 + 3 + 2 = 11$	$3 + 3 + 1 + 1 + 1 + 1 + 1 = 11$
$5 + 4 + 2 = 11$	$3 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 11$
$5 + 3 + 2 + 1 = 11$	$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 11$

$$D(11) = O(11) = 12$$

$12 = 12$	$11 + 1 = 12$
$11 + 1 = 12$	$9 + 3 = 12$
$10 + 2 = 12$	$9 + 1 + 1 + 1 = 12$
$9 + 3 = 12$	$7 + 5 = 12$
$9 + 2 + 1 = 12$	$7 + 3 + 1 + 1 = 12$
$8 + 4 = 12$	$7 + 1 + 1 + 1 + 1 + 1 = 12$
$8 + 3 + 1 = 12$	$5 + 5 + 1 + 1 = 12$
$7 + 5 = 12$	$5 + 3 + 3 + 1 = 12$
$7 + 4 + 1 = 12$	$5 + 3 + 1 + 1 + 1 + 1 = 12$
$7 + 3 + 2 = 12$	$5 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 12$
$6 + 5 + 1 = 12$	$3 + 3 + 3 + 3 = 12$
$6 + 4 + 2 = 12$	$3 + 3 + 3 + 1 + 1 + 1 = 12$
$6 + 3 + 2 + 1 = 12$	$3 + 3 + 1 + 1 + 1 + 1 + 1 + 1 = 12$
$5 + 4 + 3 = 12$	$3 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 12$
$5 + 4 + 2 + 1 = 12$	$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 12$

$$D(12) = O(12) = 15$$

$13 = 13$	$13 = 13$
$12 + 1 = 13$	$11 + 1 + 1 = 13$
$11 + 2 = 13$	$9 + 3 + 1 = 13$
$10 + 3 = 13$	$9 + 1 + 1 + 1 + 1 = 13$
$10 + 2 + 1 = 13$	$7 + 5 + 1 = 13$
$9 + 4 = 13$	$7 + 3 + 3 = 13$
$9 + 3 + 1 = 13$	$7 + 3 + 1 + 1 + 1 = 13$
$8 + 5 = 13$	$7 + 1 + 1 + 1 + 1 + 1 + 1 = 13$
$8 + 4 + 1 = 13$	$5 + 5 + 3 = 13$
$8 + 3 + 2 = 13$	$5 + 5 + 1 + 1 + 1 = 13$
$7 + 6 = 13$	$5 + 3 + 3 + 1 + 1 = 13$
$7 + 5 + 1 = 13$	$5 + 3 + 1 + 1 + 1 + 1 + 1 = 13$
$7 + 4 + 2 = 13$	$5 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 13$
$7 + 3 + 2 + 1 = 13$	$3 + 3 + 3 + 3 + 1 = 13$
$6 + 5 + 2 = 13$	$3 + 3 + 3 + 1 + 1 + 1 + 1 = 13$
$6 + 4 + 3 = 13$	$3 + 3 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 13$
$6 + 4 + 2 + 1 = 13$	$3 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 13$
$5 + 4 + 3 + 1 = 13$	$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 13$

$$D(13) = O(13) = 18$$

$14 = 14$
$13 + 1 = 14$
$12 + 2 = 14$
$11 + 3 = 14$
$11 + 2 + 1 = 14$
$10 + 4 = 14$
$10 + 3 + 1 = 14$
$9 + 5 = 14$
$9 + 4 + 1 = 14$
$9 + 3 + 2 = 14$
$8 + 6 = 14$
$8 + 5 + 1 = 14$
$8 + 4 + 2 = 14$
$8 + 3 + 2 + 1 = 14$
$7 + 6 + 1 = 14$
$7 + 5 + 2 = 14$
$7 + 4 + 3 = 14$
$7 + 4 + 2 + 1 = 14$
$6 + 5 + 3 = 14$
$6 + 5 + 2 + 1 = 14$
$6 + 4 + 3 + 1 = 14$
$5 + 4 + 3 + 2 = 14$

$13 + 1 = 14$
$11 + 3 = 14$
$11 + 1 + 1 + 1 = 14$
$9 + 5 = 14$
$9 + 3 + 1 + 1 = 14$
$9 + 1 + 1 + 1 + 1 + 1 = 14$
$7 + 7 = 14$
$7 + 5 + 1 + 1 = 14$
$7 + 3 + 3 + 1 = 14$
$7 + 3 + 1 + 1 + 1 + 1 = 14$
$7 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 14$
$5 + 5 + 3 + 1 = 14$
$5 + 5 + 1 + 1 + 1 + 1 = 14$
$5 + 3 + 3 + 3 = 14$
$5 + 3 + 3 + 1 + 1 + 1 = 14$
$5 + 3 + 1 + 1 + 1 + 1 + 1 + 1 = 14$
$5 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 14$
$3 + 3 + 3 + 3 + 1 + 1 = 14$
$3 + 3 + 3 + 1 + 1 + 1 + 1 + 1 = 14$
$3 + 3 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 14$
$3 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 14$
$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 14$

$$D(14) = O(14) = 22$$

$15 = 15$
$14 + 1 = 15$
$13 + 2 = 15$
$12 + 3 = 15$
$12 + 2 + 1 = 15$
$11 + 4 = 15$
$11 + 3 + 1 = 15$
$10 + 5 = 15$
$10 + 4 + 1 = 15$
$10 + 3 + 2 = 15$
$9 + 6 = 15$
$9 + 5 + 1 = 15$
$9 + 4 + 2 = 15$
$9 + 3 + 2 + 1 = 15$
$8 + 7 = 15$
$8 + 6 + 1 = 15$
$8 + 5 + 2 = 15$
$8 + 4 + 3 = 15$
$8 + 4 + 2 + 1 = 15$
$7 + 6 + 2 = 15$
$7 + 5 + 3 = 15$
$7 + 5 + 2 + 1 = 15$
$7 + 4 + 3 + 1 = 15$
$6 + 5 + 4 = 15$
$6 + 5 + 3 + 1 = 15$
$6 + 4 + 3 + 2 = 15$
$5 + 4 + 3 + 2 + 1 = 15$

$15 = 15$
$13 + 1 + 1 = 15$
$11 + 3 + 1 = 15$
$11 + 1 + 1 + 1 + 1 = 15$
$9 + 5 + 1 = 15$
$9 + 3 + 3 = 15$
$9 + 3 + 1 + 1 + 1 = 15$
$9 + 1 + 1 + 1 + 1 + 1 + 1 = 15$
$7 + 7 + 1 = 15$
$7 + 5 + 3 = 15$
$7 + 5 + 1 + 1 + 1 = 15$
$7 + 3 + 3 + 1 + 1 = 15$
$7 + 3 + 1 + 1 + 1 + 1 + 1 = 15$
$7 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 15$
$5 + 5 + 5 = 15$
$5 + 5 + 3 + 1 + 1 = 15$
$5 + 5 + 1 + 1 + 1 + 1 + 1 = 15$
$5 + 3 + 3 + 3 + 1 = 15$
$5 + 3 + 3 + 1 + 1 + 1 + 1 = 15$
$5 + 3 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 15$
$5 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 15$
$3 + 3 + 3 + 3 + 3 = 15$
$3 + 3 + 3 + 3 + 1 + 1 + 1 = 15$
$3 + 3 + 3 + 1 + 1 + 1 + 1 + 1 + 1 = 15$
$3 + 3 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 15$
$3 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 15$
$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 15$

$$D(15) = O(15) = 27$$

Conjecture

$D(n) = O(n)$ for **any** n?

Euler's brilliant argument

Consider the "polynomial"

$$F(x) = (1 + x^1)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5)\cdots$$

Question : What is the coefficient of x^n ?
Answer : $D(n)$.
Example :

$$\text{Coefficient of } x^5 = x^5 + x^{1+4} + x^{2+3} = D(5)$$

Consider another "polynomial"

$$G(x) = (1+x^1+x^{1+1}+x^{1+1+1}+\cdots)(1+x^3+x^{3+3}+x^{3+3+3}+\cdots)(1+x^5+x^{5+5}+x^{5+5+5}+\cdots)\cdots$$

Question : What is the coefficient of x^n ?
Answer : $O(n)$.
Example :

$$\text{Coefficient of } x^5 = x^5 + x^{1+1+3} + x^{1+1+1+1+1} = O(5)$$

We need to see why

$$F(x) \stackrel{?}{=} G(x)$$

Recall the definitions :

$$F(x) = (1 + x^1)(1 + x^2)(1 + x^3)(1 + x^4)(1 + x^5) \cdots$$
$$G(x) = (1 + x^1 + x^{1+1} + x^{1+1+1} + \cdots)(1 + x^3 + x^{3+3} + x^{3+3+3} + \cdots) \cdots$$

Inifinite geometric series

$$1 + a + a^2 + a^3 + \cdots = \frac{1}{1-a}$$

Proof :

$$S = 1 + a + a^2 + a^3 + \cdots$$

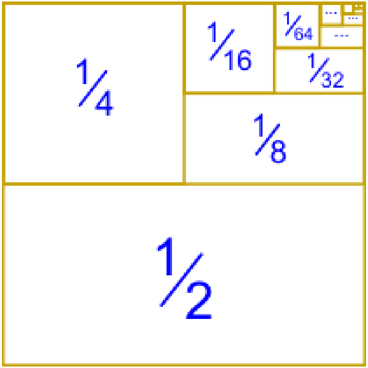
$$aS = a + a^2 + a^3 + a^4 + \cdots$$

$$(1-a)S = 1$$

$$S = \frac{1}{1-a}$$

Example :

$$1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \cdots = 2$$


1/4
1/16
1/64
1/32
1/8
1/2

Simplify $G(x)$

Recall :

$$1 + a + a^2 + a^3 + \cdots = \frac{1}{1-a}$$

Consequence :

$$G(x) = \frac{1}{1-x^1} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdots$$

Goal : $F(x) = G(x)$, or equivalently :

$$(1+x^1)(1+x^2)(1+x^3)(1+x^4)\cdots \stackrel{?}{=} \frac{1}{1-x^1}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdots$$

$$
\begin{aligned}
G(x) &= \frac{1}{1-x^1}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdots \\
&= \frac{(1-x^2)(1-x^4)(1-x^6)(1-x^8)\cdots}{(1-x^1)(1-x^2)(1-x^3)(1-x^4)(1-x^5)\cdots} \\
&= \frac{(1-x^1)(1+x^1)(1-x^2)(1+x^2)(1-x^3)(1+x^3)(1-x^4)(1+x^4)\cdots}{(1-x^1)(1-x^2)(1-x^3)(1-x^4)(1-x^5)\cdots} \\
&= (1+x^1)(1+x^2)(1+x^3)(1+x^4)\cdots \\
&= F(x)
\end{aligned}
$$

Compare the coefficients of $F(x)$ and $G(x) \Rightarrow O(n) = D(n)$!

A challenge

We reformulate Euler's discovery as follows :
$O(n) =$ number of ways to break n into a sum of numbers that are not divisible by 2.
$D(n) =$ number of ways to break n into a sum of numbers that do not repeat 2 times.

Challenge

For any natural number n and k, the number of ways of expressing n as a sum of numbers that are not divisible by k is equal to the number of ways of expressing n into a sum of numbers that do not repeat k times.

Try first an example $n = 6, k = 3$.

Basel problem

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = ?$$

A question posed by Pietro Mengoli in 1650, answered by Leonhard Euler in 1735, and named after Basel (hometown of Euler).

$$\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{5^2} = 1.46361111111111...$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{10^2} = 1.54976773116654...$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{100^2} = 1.634983900184892..$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{1000^2} = 1.64393456668155...$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{10000^2} = 1.64483407184805...$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{100000^2} = 1.64492406689822...$$

$$\vdots$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots = 1.644934066848226...$$

The guess

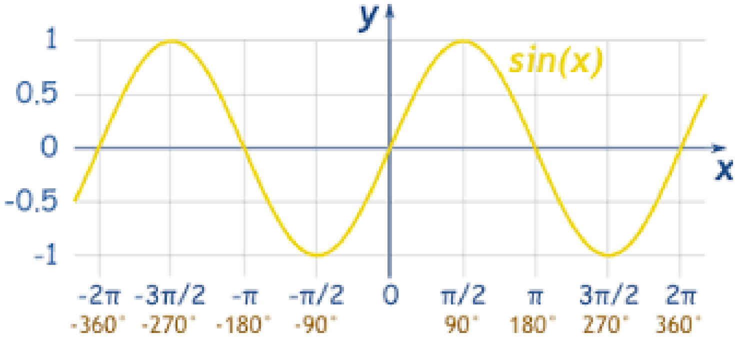
$$\sqrt{1.644934066848226\ldots \times 6}$$

$$=3.14159265358979323846264338327950288419716939937510$$
$$58209749445923078164062862089986280348253421170679 82$$
$$1480865132823066470938446\ldots$$

Conjecture

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots \overset{?}{=} \frac{\pi^2}{6}$$

Sine function



Reproducing a "polynomial" from its zeros :

$$\sin(x) = x(1 - \frac{x}{\pi})(1 + \frac{x}{\pi})(1 - \frac{x}{2\pi})(1 + \frac{x}{2\pi})(1 - \frac{x}{3\pi})(1 + \frac{x}{3\pi}) \cdots$$

$$\sin(x)/x = 1 - \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \cdots \right) x^2 + \cdots$$

Approximating $\sin(x)/x$:

$$\sin(x)/x = 1 - \frac{x^2}{6} + \frac{x^4}{120} + \cdots$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

Happy π-Day !